Extremal and Probabilistic Graph Theory
Lecture 8

March 24th, Thursday

Today we prove a lower bound of $ex_r(n, K_s^{(r)})$ mentioned in Lecture 7.

Theorem 8.1.
$$ex_r(n, K_s^{(r)}) \geqslant \left(1 - \left(\frac{r-1}{s-1}\right)^{r-1} + o(1)\right) \cdot \binom{n}{r}.$$

Proof. We need to show there exists a K_s^r-free n-vertex r-graph G with at least

$$\left(1 - \left(\frac{r-1}{s-1}\right)^{r-1} + o(1)\right)\binom{n}{r}$$

many edges. Equivalently, we can construct a complement graph $H = G^c$ satisfying:

(i) $e(H) \leqslant \left(\left(\frac{r-1}{s-1}\right)^{r-1} + o(1)\right)\binom{n}{r}$;

(ii) Any s-subset of $V(H)$ contains at least one edge.

The graph H is constructed as follows. Let H be a graph with $V(H) = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_{s-1}$, where $\lfloor \frac{n}{s-1} \rfloor \leqslant |V_1| \leqslant \cdots \leqslant |V_{s-1}| \leqslant \lceil \frac{n}{s-1} \rceil$. For each $e \in \binom{V}{r}$, e is an edge if and only if there exists an index j such that
$$\sum_{i=0}^{k-1} |e \cap V_{j+i}| \geqslant k+1 \tag{8.1}$$
for $\forall 1 \leqslant k \leqslant r-1$. (Note: The subscripts of V_{j+i} are taken module $s-1$.)

We first verify (ii) for H. In fact, we can show something stronger: for any $S \in \binom{V}{s}$, then there exists an index j such that
$$\sum_{i=0}^{k-1} |S \cap V_{j+i}| \geqslant k+1$$
for $\forall 1 \leqslant k \leqslant s-1$.

We shall see that this condition implies that there exists $e \in \binom{S}{r}$ satisfying (8.1). The proof of the above statement is left as an exercise. (Or see a proof in the end of this notes.)

All we need is to verify (i) for H. It is easier to count the number of ordered r-tuple $(x_1, x_2, \cdots, x_r)$ such that $\{x_1, x_2, \cdots, x_r\}$ is an edge. Let K be the number of such ordered r-tuples in H. Then $K = r! \cdot |E(H)|$. So we need to show

$$\begin{aligned} K &\leqslant \left(r! \cdot \left(\frac{r-1}{s-1} \right)^{r-1} + o(1) \right) \binom{n}{r} \\ &\approx \left(\frac{r-1}{s-1} \right)^{r-1} n^r + o(n^r). \end{aligned}$$

Definition: For an edge $\{x_1, \cdots x_r\}$, we define the *signature* of the r-tuple $(x_1, x_2, \cdots, x_r)$ as $(j, \vec{c})$ such that

- j is the index satisfying (8.1);
- $\vec{c} = (c_1, c_2, \cdots, c_r)$ where $x_i \in V_{j+t}$ iff $c_i = t$. We see that $0 \leqslant c_i \leqslant r-2$.

Observation 1: The signature of an r-tuple $(x_1, x_2, \cdots, x_r)$ determines whether it forms an edge or not.

Definition: We say $\vec{c} = (c_1, c_2, \cdots, c_r)$ is *legal* if any r-tuple with signature $(j, \vec{c})$ forms an edge.

Observation 2: Each legal $\vec{c}$ has approximately $(s-1)\left(\frac{n}{s-1}\right)^r$ r-tuples whose signature is $(j, \vec{c})$ for some $1 \leqslant j \leqslant s-1$.

Therefore, it suffices to show there are at most

$$\frac{\left(\frac{r-1}{s-1}\right)^{r-1} n^r}{(s-1)\left(\frac{n}{s-1}\right)^r} = (r-1)^{r-1}$$

legal vectors $\vec{c} = (c_1, c_2, \cdots, c_r)$, where $0 \leqslant c_i \leqslant r-2$. Denote $\mathcal{C} = \{(c_1, c_2, \cdots, c_r) : 0 \leqslant c_i \leqslant r-2\}$ and $L =$ the subset of legal vectors in $\mathcal{C}$. Then it is equivalent to show

$$\frac{|L|}{|\mathcal{C}|} \leqslant \frac{(r-1)^{r-1}}{(r-1)^r} = \frac{1}{r-1}.$$

We will partition $\mathcal{C}$ into many many subfamilies as follows, each of which has $r-1$ vectors and has at most 1 legal vector $\vec{c}$. For each $\vec{c} = (c_1, c_2, \cdots, c_r)$, define

$$\text{shift}(\vec{c}, l) = \vec{c} + l \cdot \vec{1} = (c_1 + l, c_2 + l, \cdots, c_r + l)$$

for $0 \leqslant l \leqslant r-2$. Here $c_i + l$ are taken under $\mathbb{Z}_{r-1} = \{0, 1, 2, \cdots, r-2\}$.

Consider the subfamilies $\{\text{shift}(\vec{c}, l) : 0 \leqslant l \leqslant r-2\}$, which partition $\mathcal{C}$.

We are left to show there are at most 1 legal vector in every subfamily $\{\text{shift}(\vec{c}, l)\}$. Suppose for contradictions that there are 2 legal members, say without loss of generality

$$\begin{cases} \text{shift}(\vec{c}, 0) = \vec{c}; \\ \text{shift}(\vec{c}, -l). \end{cases}$$

Let r-tuple $(x_1, x_2, \cdots, x_r)$ have signature $(j, \vec{c})$, where $\{x_1, x_2, \cdots, x_r\}$ is an edge of H. Consider another r-tuple $f = (x'_1, ..., x'_r)$, where each x'_i is obtained by cyclically "shifting" x_i from its own cluster to the l^{th} cluster on its left (here, we view clusters $V_j, ..., V_{j+r-2}$ are cyclically arranged). We point out that the signature of f is $(j, \text{shift}(\vec{c}, -l))$.

Let $e = \{x_1, \cdots, x_r\}$ and $a_{j+i} = e \cap V_{j+i}$ for $i \in \mathbb{Z}_{r-1} = \{0, 1, \cdots, r-2\}$. Taking $k = l$ from (8.1), we have

$$\sum_{i=0}^{l-1} |e \cap V_{j+i}| \geqslant l + 1,$$

i.e.

$$a_j + a_{j+1} + \cdots + a_{j+l-1} \geqslant l + 1. \tag{8.2}$$

Let $a'_{j+i} = f \cap V_{j+i}$ for $i \in \mathbb{Z}_{r-1}$. Then we see that $a'_{j+i} = a_{j+i+l}$, where $i + l$ is taken under $\mathbb{Z}_{r-1}$. By taking $k = r - l - 1$ from (8.1) for f, we get

$$\sum_{i=0}^{r-l-2} |f \cap V_{j+i}| = a'_j + ... + a'_{j+r-l-2} = a_{j+l} + \cdots + a_{j+r-2} \geqslant r - l. \tag{8.3}$$

Adding up (8.2) and (8.3), we have

$$r = a_j + a_{j+1} + \cdots + a_{j+r-2} \geqslant r + 1.$$

This contradiction finishes the proof. ∎

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Another Proof. (Contributed by X. Yuan)

Make the same construction with proof 1. We prove a lemma:

Lemma: *For non-negative integers $v_1, ..., v_{t-1}$, $v_1 + \cdots + v_{t-1} = t$. Let*

$$T_j = (t-1)v_j + (t-2)v_{j+1} + \cdots + v_{j+t-2},$$

then there is a unique j_0 such that $T_{j_0} = \max_{i=1,\cdots,t-1} \{T_i\}$ and

$$\sum_{i=0}^{k-1} v_{j+i} \geq k + 1 \text{ for } \forall 1 \leq k \leq t - 1 \tag{8.4}$$

holds if and only if $j = j_0$. (Here the subscripts of v_{j+i} are taken module $t - 1$.)

Proof of lemma.

Say $j_0 = 1$, then for $j \neq 1$

$$T_1 \geq T_j \Rightarrow (t-1)(v_1 + \cdots + v_{j-1}) \geq (j-1)t \Rightarrow v_1 + \cdots v_{j-1} \geq j \tag{8.5}$$

$$\Rightarrow (t-1)(v_1 + \cdots + v_{j-1}) \geq (t-1)j > (j-1)t \Rightarrow T_1 > T_j \tag{8.6}$$

From (8.6) we know that the maximum is unique. (*Remark.* In fact, $\{v_1, ..., v_{t-1}\}$ are distinctive since s and $s - 1$ are coprime.) On the other hand,

$$T_1 < T_j \Rightarrow v_1 + \cdots v_{j-1} < j$$

doesn't satisfy (8.4), together with (8.5) we have (8.4) holds if and only if $j = 1$. This finishes the lemma.

From the lemma we know that (ii) holds since for $t = s$, the maximum must exist.

For (i), since the notations and some inequalities are the same with proof 1, here we only show that there are at most $(r-1)^{r-1}$ legal vectors $\vec{c} = (c_1, c_2, \cdots, c_r)$.

Let $d_i = \#\{j | c_j = i-1, j = 1, 2, ..., r\}$, $i = 1, ..., r-1$. Then $d_1 + \cdots + d_{r-1} = r$. From (8.1) we have

$$\sum_{i=0}^{k-1} d_{j+i} \geq k+1 \text{ for } \forall 1 \leq k \leq r-1.$$

(Here the subscripts of d_{j+i} are taken module $r-1$.) From the lemma, there is at most 1 permutation of $(c_1, ..., c_r)$ which is legal. Thus

$$\frac{|L|}{|\mathcal{C}|} \leqslant \frac{(r-1)^{r-1}}{(r-1)^r} = \frac{1}{r-1}.$$

This finishes the proof. ∎

Remark. In fact, the map from 'signature's to r-subsets is surjective but not injective, i.e. one r-subset can have more than one 'signature'. But here, from the proof we can see, the map from 'legal signature's to edges is bijective. Amazing construction!